GROWTH OF TRANSVERSELY HOMOGENEOUS FOLIATIONS

JESÚS A. ÁLVAREZ LÓPEZ AND ROBERT WOLAK

ABSTRACT. For transversely homogeneous foliations on compact manifolds satisfying mild conditions, it is shown that either all holonomy covers of the leaves have polynomial growth with degree bounded by a common constant, or all holonomy covers of the leaves have exponential growth. This is an extension to transversely homogeneous foliations of a recent answer given by Breuillard-Gelander to a question of Carrière.

1. INTRODUCTION

Let G be a Lie group, and $P \subset G$ a closed subgroup. The concept of (transversely homogeneous) G/P-foliation was introduced by Blumenthal [4]: it is a smooth foliation with the structure given by a defining cocycle with values in the G-manifold G/P.

Assume that the G-action on G/P is faithful, and G/P is connected. Let $\mathcal{F}$ be a G/P-foliation on a compact connected manifold M. Then Blumenthal proved that, if $\pi_1(M)$ has non-exponential growth (respectively, polynomial growth of degree d), then all leaves of $\mathcal{F}$ have non-exponential growth (respectively, polynomial growth of degree d) [4, Theorem 1]. (Recall that the growth is one of the classical invariants of leaves of foliations on compact manifolds.)

When $\mathcal{F}$ is a Lie G-foliation ($P = \{1\}$), Carrière has shown that [7]:

- the leaves of $\mathcal{F}$ are Følner if and only if G is solvable;
- the leaves of $\mathcal{F}$ have polynomial growth if and only if G is nilpotent; and,
- in the last case, the leaves of $\mathcal{F}$ have polynomial growth with degree less or equal than the degree of nilpotence of G.

Carrière asked in [7] about the existence of a Lie G-foliation on a compact manifold whose leaves have neither polynomial nor exponential growth. This question was recently answered by Breuillard-Gelander [5, Theorem 10.1], obtaining the following dichotomy as consequence of their study of a topological Tits alternative:

- either all leaves of $\mathcal{F}$ have polynomial growth with degree bounded by a common constant;
- or all leaves of $\mathcal{F}$ have exponential growth.

Indeed, Carrière and Breuillard-Gelander stated these results for Riemannian foliations on compact manifolds, either by considering the residual set of leaves without holonomy, or by considering the holonomy covers of all leaves. This kind of extension is straightforward by Molino's theory [15].

In this paper, we show that the arguments of Breuillard-Gelander about growth of Lie foliations can be extended to G/P-foliations assuming some mild condition. Let us remark that a G/P-foliation may not be Riemannian (it is Riemannian if P is compact). Precisely, we prove the following complement of Blumenthal's observations about growth of transversely homogeneous foliations.

Theorem 1.1. *Let G be a Lie group, and $P \subset G$ a closed subgroup such that G/P is connected and the G-action on G/P is faithful. Let $\mathcal{F}$ be a G/P-foliation on a compact connected manifold. Then:*

- *either all holonomy covers of the leaves of $\mathcal{F}$ have polynomial growth with degree bounded by a common constant;*
- *or all non-compact holonomy covers of the leaves have exponential growth.*

A smooth foliation is called taut when there is a Riemannian metric on the ambient manifold so that the leaves have zero mean curvature. By the global stability theorem of Pereira [16, Theorem 1 and Remark 6] (see also [19]), we get the following consequence from Theorem 1.1.

Corollary 1.2. *Let G be a Lie group, and $P \subset G$ a closed subgroup such that G/P is connected and the G-action on G/P is faithful. Let $\mathcal{F}$ be a taut on a compact connected manifold. Then:*

- *either all holonomy covers of the leaves of $\mathcal{F}$ have polynomial growth with degree bounded by a common constant;*
- *or all holonomy covers of the leaves have exponential growth.*

2. Preliminaries

2.1. **Coarse quasi-isometries and growth of metric spaces.** A *net* in a metric space M, with metric d, is a subset $A \subset M$ that satisfies $d(x, A) \leq C$ for some $C > 0$ and all $x \in M$; the term C-*net* is also used. A *coarse quasi-isometry* between M and another metric space M' is a bi-Lipschitz bijection between nets of M and M'; in this case, M and M' are said to be *coarsely quasi-isometric* (in the sense of Gromov) [9]. If such a bi-Lipschitz bijection, as well as its inverse, has dilation $\leq \lambda$, and it is defined between C-nets, then it will be said that the coarse quasi-isometry has *distortion* (C, λ). A family of coarse quasi-isometries with a common distortion will be called *uniform*, and the corresponding metric spaces are called *uniformly* coarsely quasi-isometric.

The version of growth for metric spaces given here is taken from [1]. Since [1] is not finished, some short proofs are included.

Recall that, given non-decreasing functions[1] $u, v : [0, \infty) \to [0, \infty)$, it is said that u is *dominated* by v, written $u \preccurlyeq v$, when there are $a, b \geq 1$ and $c \geq 0$ such that $u(r) \leq a\, v(br)$ for all $r \geq c$. If $u \preccurlyeq v \preccurlyeq u$, then it is said that u and v represent the same *growth type*; this is an equivalence relation and "$\preccurlyeq$" defines a partial order relation between growth types called *domination*. For a family of pairs of non-decreasing functions $[0, \infty) \to [0, \infty)$, *uniform* domination means that those pairs satisfy the above condition of domination with the same constants a, b, c. A family of functions $[0, \infty) \to [0, \infty)$ will be said to have *uniformly* the same growth type if they uniformly dominate one another.

For a complete connected Riemannian manifold L, the growth type of each mapping $r \mapsto \mathrm{vol}\, B(x, r)$ is independent of x and is called the *growth type* of L. Another definition of *growth type* can be similarly given for metric spaces whose bounded sets are finite, where the number of points is used instead of the volume.

Let M be a metric space with metric d. A *quasi-lattice* Γ of M is a C-net of M for some $C \geq 0$ such that, for every $r \geq 0$, there is some $K_r \geq 0$ such that $\mathrm{card}(\Gamma \cap B(x, r)) \leq K_r$ for every $x \in M$. It is said that M is of *coarse bounded geometry* if it has a quasi-lattice. In this case, the *growth type* of M can be defined as the growth type of any quasi-lattice Γ of M; i.e., it is the growth type of the *growth function* $r \mapsto v_\Gamma(x, r) = \mathrm{card}(B(x, r) \cap \Gamma)$ for any $x \in \Gamma$. This definition can be proved to be independent of Γ as follows. Let Γ' be another quasi-lattice in M. So Γ and Γ' are C-nets in M for some $C \geq 0$, and there is some $K_r \geq 0$ for each $r \geq 0$ such that $\mathrm{card}(B(x, r) \cap \Gamma) \leq K_r$ and $\mathrm{card}(B(x, r) \cap \Gamma') \leq K_r$ for all $x \in M$. Fix points $x \in \Gamma$ and $x' \in \Gamma'$, and let $\delta = d(x, x')$. Because $B(x, r) \subset B(x', r + \delta)$ and Γ' is a C-net, it follows that

$$B(x, r) \cap \Gamma \subset \bigcup_{y' \in B(x', r + \delta + C) \cap \Gamma'} B(y', C) \cap \Gamma' \, ,$$

yielding

$$v_\Gamma(x, r) \leq K_C\, v_{\Gamma'}(x', r + \delta + C) \leq K_C\, v_{\Gamma'}(x', (1 + \delta + C)r)$$

for all $r \geq 1$. Hence the growth type of $r \mapsto v_\Gamma(x, r)$ is dominated by the growth type of $r \mapsto v_{\Gamma'}(x, r)$.

For a family of metric spaces, if they satisfy the above condition of coarse bounded geometry with the same constants C and K_r, then they are said to have *uniformly* coarse bounded geometry. If moreover the lattices involved in this condition have growth functions defining uniformly the same growth type, then these metric spaces are said to have *uniformly* the same growth type.

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[1] Usually, growth types are defined by using non-decresing functions $\mathbb{Z}^+ \to [0, \infty)$, but this gives rise to an equivalent concept.

The condition of coarse bounded geometry is satisfied by complete connected Riemannian manifolds of bounded geometry, and by discrete metric spaces with a uniform upper bound on the number of points in all balls of each given radius [3]. In those cases, the two given definitions of growth type are equal.

Lemma 2.1 (Álvarez López-Candel [2])**.** *Two coarsely quasi-isometric metric spaces of coarse bounded geometry have the same growth type. Moreover, if a family of metric spaces are uniformly coarsely quasi-isometric to each other, then they have uniformly the same growth type.*

Proof. Let $\phi : A \to A'$ be a coarse quasi-isometry between metric spaces M and M' of coarse bounded geometry. Then A is of coarse bounded geometry too, and thus it has some lattice Γ, which is also a lattice in M because A is a net. Since ϕ is a bi-lipschitz bijection, it easily follows that Γ and $\phi(\Gamma)$ have the same growth type, and that $\phi(\Gamma)$ is a lattice in A', and thus in M' too because A' is a net. This argument has an obvious uniform version for a family of metric spaces. □

2.2. **Pseudogroups.** A *pseudogroup* (*of local transformations*) on a topological space T is a collection $\mathcal{H}$ of homeomorphisms between open subsets of T that contains the identity map and is closed under composition (wherever defined), inversion, restriction and combination (union) of maps. Such a pseudogroup $\mathcal{H}$ is *generated* by a set $E \subset \mathcal{H}$ if every element of $\mathcal{H}$ can be obtained from E by using the above pseudogroup operations; usually, E will be symmetric ($h^{-1} \in E$ if $h \in E$). The *orbit* of a point $x \in T$ is the set $\mathcal{H}(x) = \{\, h(x) \mid h \in \mathcal{H}, \ x \in \operatorname{dom} h \,\}$. Many other basic dynamical concepts can be generalized to pseudogroups because they are natural generalizations of dynamical systems (each group action on a topological space generates a pseudogroup). The *restriction* of $\mathcal{H}$ to an open subset $U \subset T$ is the pseudogroup $\mathcal{H}|_U = \{\, h \in \mathcal{H} \mid \operatorname{dom} h \cup \operatorname{im} h \subset U \,\}$. It is said that $\mathcal{H}$ is *quasi-analytic* when any $h \in \mathcal{H}$ is the identity around any $x \in \operatorname{dom} h$ if h is the identity on some open set whose closure contains x; *quasi-analytic* group actions are similarly defined, which means that they generate quasi-analytic pseudogroups. Pseudogroups that are equivalent in the following sense should be considered to have the same dynamics.

Definition 2.2 (Haefliger [11, 12])**.** Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups on respective topological spaces T and T'. An *equivalence* $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal collection Φ of homeomorphisms of open subsets of T to open subsets of T' such that:

- If $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h'\phi h \in \Phi$; and
- $\mathcal{H}$ and $\mathcal{H}'$ are generated by the maps of the form $\psi^{-1}\phi$ and $\psi\phi^{-1}$, respectively, with $\phi, \psi \in \Phi$.

In this case, $\mathcal{H}$ and $\mathcal{H}'$ are said to be *equivalent*.

If $\Phi : \mathcal{H} \to \mathcal{H}'$ is an equivalence, then $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$ is an equivalence $\mathcal{H}' \to \mathcal{H}$, called the *inverse* of Φ. An equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$ is *generated* by a subset $\Phi_0 \subset \Phi$ if all of the elements of Φ can be obtained by restriction and combination of composites $h'\phi h$ with $h \in \mathcal{H}$, $\phi \in \Phi_0$ and $h' \in \mathcal{H}'$. If $\Phi : \mathcal{H} \to \mathcal{H}'$ and $\Psi : \mathcal{H}' \to \mathcal{H}''$ are equivalences, then the maps $\psi\phi$, for $\phi \in \Phi$ and $\psi \in \Psi$, generate an equivalence $\Psi\Phi : \mathcal{H} \to \mathcal{H}'$, called the *composite* of Φ and Ψ. Thus the equivalence of pseudogroups is an equivalence relation. An equivalence $\Phi : \mathcal{H} \to \mathcal{H}'$ induces a homeomorphism between the corresponding orbit spaces, $\bar{\Phi} : T/\mathcal{H} \to T'/\mathcal{H}'$.

A basic example of a pseudogroup equivalence is the following. Let $\mathcal{H}$ be a pseudogroup of local transformations of a space T, let $U \subset T$ be an open subset that meets every $\mathcal{H}$-orbit. Then the inclusion map $U \hookrightarrow T$ generates an equivalence $\mathcal{H}|_U \to \mathcal{H}$. In fact, this example can be used to describe any pseudogroup equivalence in the following way. Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups of local transformations of respective spaces T and T', and let $\Phi : \mathcal{H} \to \mathcal{H}'$ be an equivalence. Let $\mathcal{H}''$ be the pseudogroup of local transformations of $T'' = T \sqcup T'$ generated by $\mathcal{H} \cup \mathcal{H}' \cup \Phi$. Then the inclusions of T and T' in T'' generate equivalences $\Psi_1 : \mathcal{H} \to \mathcal{H}''$ and $\Psi_2 : \mathcal{H}' \to \mathcal{H}''$ so that $\Phi = \Psi_2^{-1}\Psi_1$.

For a pseudogroup $\mathcal{H}$ on a locally compact space T, the orbit space $T/\mathcal{H}$ is compact if and only if there exists a relatively compact open subset of T that meets every $\mathcal{H}$-orbit. The following is a stronger "compactness" condition on a pseudogroup.

Definition 2.3 (Haefliger [11])**.** Let $\mathcal{H}$ be a pseudogroup on a locally compact space T. It is said that $\mathcal{H}$ is *compactly generated* if there is a relatively compact open set U in T meeting each $\mathcal{H}$-orbit, and such that $\mathcal{H}|_U$ is generated by a finite symmetric collection E so that each $g \in E$ has an extension $\bar{g} \in \mathcal{H}$ with $\overline{\operatorname{dom} g} \subset \operatorname{dom} \bar{g}$.

It was observed in [11] that this notion is invariant by equivalences, and that the relatively compact open set U meeting each orbit can be chosen arbitrarily. If E satisfies the conditions of Definition 2.3, it will be called a *system of compact generation* of $\mathcal{H}$ on U.

For $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$, let $\operatorname{germ}(h, x)$ denote the germ of h at x. Then

$$\mathfrak{H} = \{\operatorname{germ}(h, x) \mid h \in \mathcal{H}, \ x \in \operatorname{dom} h\}$$

becomes a topological groupoid, with object space T, equipped with the étale topology, the operation induced by composition, and the source and target projections to T. For $x, y \in T$, let $\mathfrak{H}_x$ (respectively, $\mathfrak{H}^y$) denote the set of elements in $\mathfrak{H}$ with source x (respectively, with target y), and let $\mathfrak{H}_x^y = \mathfrak{H}_x \cap \mathfrak{H}^y$; in particular, the group $\mathfrak{H}_x^x$ will be called the *germ group* of $\mathcal{H}$ at x. Points in the same $\mathcal{H}$-orbit have isomorphic germ groups (if $y \in \mathcal{H}(x)$, an isomorphism $\mathfrak{H}_y^y \to \mathfrak{H}_x^x$ is given by conjugation with any element in $\mathfrak{H}_x^y$); hence the germ groups of the orbits make sense up to isomorphism. Under pseudogroup equivalences, corresponding orbits have isomorphic germ

groups. The set $\mathfrak{H}_x$ will be called the *germ cover* of the orbit $\mathcal{H}(x)$ with base point x. The target map restricts to a surjective map $\mathfrak{H}_x \to \mathcal{H}(x)$ whose fibers are bijective to $\mathfrak{H}_x^x$ (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{H}_x^x \to \mathfrak{H}_x^y$ is given by left product with any element in $\mathfrak{H}_x^y$); thus $\mathfrak{H}_x$ is finite if and only if both $\mathfrak{H}_x^x$ and $\mathcal{H}(x)$ are finite. Moreover germ covers based on points in the same orbit are also bijective (if $y \in \mathcal{H}(x)$, a bijection $\mathfrak{H}_y \to \mathfrak{H}_x$ is given by right product with any element in $\mathfrak{H}_x^y$); therefore the germ covers of the orbits make sense up to bijections.

2.3. **Quasi-isometry type of orbits.** Let $\mathcal{H}$ be a pseudogroup on a space T, and E a symmetric set of generators of $\mathcal{H}$. For each $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$, let $|h|_{E,x}$ be the length of the shortest expression of $\operatorname{germ}(h,x)$ as product of germs of maps in E (being 0 if $\operatorname{germ}(h,x) = \operatorname{germ}(\operatorname{id}_T, x)$). For each $x \in T$, define metrics d_E on $\mathcal{H}(x)$ and $\mathfrak{H}_x$ by

$$d_E(y, z) = \min\{ |h|_{E,x} \mid h \in \mathcal{H}, \ y \in \operatorname{dom} h, \ h(y) = z \} ,$$

$$d_E(\operatorname{germ}(f, x), \operatorname{germ}(g, x)) = |fg^{-1}|_{E, g(x)} .$$

Notice that

$$d_E(f(x), g(x)) \leq d_E(\operatorname{germ}(f, x), \operatorname{germ}(g, x)) .$$

Moreover, on the germ covers, d_E is right invariant in the sense that, if $y \in \mathcal{H}(x)$, the bijection $\mathfrak{H}_y \to \mathfrak{H}_x$, given by right multiplication with any element in $\mathfrak{H}_x^y$, is isometric; so the isometry types of the germ covers of the orbits make sense without any reference to base points. In fact, the definition of d_E on $\mathfrak{H}_x$ is analogous to the right invariant metric d_S on a group Γ defined by a symmetric system of generators S: $d_S(\gamma, \delta) = |\gamma \delta^{-1}|$ for $\gamma, \delta \in \Gamma$, where $|\gamma|$ is the length of the shortest expression of γ as product of elements of S (being 0 if $\gamma = 1$).

Assume that $\mathcal{H}$ is compactly generated and T locally compact. Let $U \subset T$ be a relatively compact open subset that meets all $\mathcal{H}$-orbits, let $\mathcal{G} = \mathcal{H}|_U$, and let E be a symmetric system of compact generation of $\mathcal{H}$ on U. With this conditions, the quasi-isometry type of the $\mathcal{G}$-orbits with d_E may depend on E [2, Section 6]. So the following additional condition on E is considered.

Definition 2.4 (Álvarez López-Candel [2, Definition 4.2]). With the above notation, it is said that E is *recurrent* if, for any relatively compact open subset $V \subset U$ that meets all $\mathcal{G}$-orbits, there exists some $R > 0$ such that $\mathcal{G}(x) \cap V$ is an R-net in $\mathcal{G}(x)$ with d_E for all $x \in U$.

The role played by V in Definition 2.4 can actually be played by any relatively compact open subset that meets all orbits [2, Lemma 4.3]. Furthermore there always exists a recurrent system of compact generation on U [2, Corollary 4.5].

In other words, the recurrence of E means that there is some $N \in \mathbb{N}$ such that

$$U = \bigcup_{h \in E^N} h^{-1}(V \cap \operatorname{im} h) , \tag{1}$$

where E^N is the family of compositions of at most N elements of E.

Theorem 2.5 (Álvarez López-Candel [2, Theorem 4.6])**.** *Let $\mathcal{H}$ and $\mathcal{H}'$ be compactly generated pseudogroups on locally compact spaces T and T', let U and U' be relatively compact open subsets of T and T' that meet all orbits of $\mathcal{H}$ and $\mathcal{H}'$, let $\mathcal{G}$ and $\mathcal{G}'$ denote the restrictions of $\mathcal{H}$ and $\mathcal{H}'$ to U and U', and let E and E' be recurrent symmetric systems of compact generation of $\mathcal{H}$ and $\mathcal{H}'$ on U and U', respectively. Suppose that there exists an equivalence $\mathcal{H} \to \mathcal{H}'$, and consider the induced equivalence $\mathcal{G} \to \mathcal{G}'$ and homeomorphism $U/\mathcal{G} \to U'/\mathcal{G}'$. Then the $\mathcal{G}$-orbits with d_E are uniformly quasi-isometric to the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

An obvious modification of the arguments of the proof of [2, Theorem 4.6] gives the following.

Theorem 2.6. *With the notation and conditions of Theorem 2.5, the germ covers of the $\mathcal{G}$-orbits with d_E are uniformly quasi-isometric to the germ covers of the corresponding $\mathcal{G}'$-orbits with $d_{E'}$.*

Corollary 2.7. *With the notation and conditions of Theorems 2.5 and 2.6, the corresponding orbits of $\mathcal{G}$ and $\mathcal{G}'$, as well as their germ covers, have the same growth type, uniformly.*

Proof. This follows from Lemma 2.1 and Theorems 2.5 and 2.6. $\qquad\square$

2.4. **Growth of leaves.** Let us recall some basic concepts of foliation theory (see e.g. [13, 14, 8, 6, 20]). Let $\mathcal{F}$ be a smooth foliation on a manifold M given by a defining cocycle (U_i, p_i, h_{ij}) [11, 12], where $p_i : U_i \to T_i$, and $h_{ij} : p_i(U_i \cap U_j) \to p_j(U_i \cap U_j)$ is determined by the condition $p_j = h_{ij}p_i$ on $U_i \cap U_j$. We can assume that (U_i, p_i, h_{ij}) is induced by a regular foliation atlas: the sets U_i are the domains of the foliation charts, the maps p_i are the local projections whose fibers are the plaques, and the maps h_{ij} are the transverse components of the changes of coordinates. The equivalence class of the pseudogroup $\mathcal{H}$ on $T = \bigsqcup T_i$ generated by the maps h_{ij} is independent of the choice of defining cocycle, and is called the *holonomy pseudogroup* of $\mathcal{F}$. There is a canonical identity between the space of leaves and the space of $\mathcal{H}$-orbits, $M/\mathcal{F} \equiv T/\mathcal{H}$. The *holonomy group* of each leaf L is defined as the germ group of the corresponding orbit. It can be considered as a quotient of the fundamental group of L by taking "chains" of sets U_i along loops in L, and the corresponding covering space is called the *holonomy cover* $\widetilde{L}$ of L. If $\mathcal{F}$ admits a countable defining cocycle, then the leaves in some saturated residual subset of M have trivial holonomy groups [13, 14], and therefore they can be identified to their holonomy covers.

Suppose that M is compact. Then, given any Riemannian metric g on M, for each leaf L, the differentiable (and coarse) quasi-isometry types of $g|_L$ and its lift to $\widetilde{L}$ are independent of the choice of g; they depend only on $\mathcal{F}$ and L. On the other hand, $\mathcal{H}$ is compactly generated [11], which can be seen as follows. There is some defining cocycle (U'_i, p'_i, h'_{ij}), with $p'_i : U'_i \to T'_i$,

such that $\overline{U_i} \subset U_i'$, $T_i \subset T_i'$ and p_i' extends p_i. Therefore each h_{ij}' is an extension of h_{ij} so that $\overline{\operatorname{dom} h_{ij}} \subset \operatorname{dom} h_{ij}'$. Moreover $\mathcal{H}$ is the restriction to T of the pseudogroup $\mathcal{H}'$ on $T' = \bigsqcup_i T_i'$ generated by the maps h_{ij}', and T is a relatively compact open subset of T' that meets all $\mathcal{H}'$-orbits.

Let E denote the collection of maps h_{ij}. According to Theorems 2.5 and 2.6, by using the above T' and $\mathcal{H}'$, it follows that the quasi-isometry type of the $\mathcal{H}$-orbits and their germ covers with d_E are independent of the choice of (U_i, p_i, h_{ij}) under the above conditions; in fact, they are coarsely quasi-isometric to the corresponding leaves, and therefore they have the same growth type [7] (this is an easy consequence of the existence of a uniform bound of the diameter of the plaques). Similarly, the germ covers of the $\mathcal{H}$-orbits are also quasi-isometric to the holonomy covers of the corresponding leaves.

3. Growth of homogeneous pseudogroups

Let G be a Lie group, and $P \subset G$ a closed subgroup such that the G-action on G/P is faithful; thus this action is also quasi-analytic by the analyticity of G/P. Let us define a (*homogeneous*) G/P-*pseudogroup* as a pseudogroup equivalent to the pseudogroup $\mathcal{H}$ generated by the action of some subgroup $\Gamma \subset G$ on some Γ-invariant open subset $T \subset G/P$. Suppose that $\mathcal{H}$ is compactly generated, and let $\mathcal{G} = \mathcal{H}|_U$ for some relatively compact open subset $U \subset T$ that meets all $\mathcal{H}$-orbits. For every $\gamma \in \Gamma$ with $\gamma \cdot U \cap U \neq \emptyset$, let h_γ denote the restriction $U \cap \gamma^{-1} \cdot U \to \gamma \cdot U \cap U$ of the left translation $\gamma \cdot : G \to G$. There is a finite symmetric set $S = \{s_1, \ldots, s_k\} \subset \Gamma$ such that $E = \{h_{s_1}, \ldots, h_{s_k}\}$ is a recurrent system of compact generation of $\mathcal{H}$ on U; in fact, by reducing Γ if necessary, we can assume that S generates Γ. For each $x \in U$, let
$$\Gamma_{U,x} = \{\, \gamma \in \Gamma \mid \gamma \cdot x \in U \,\} \,.$$
Let $\mathfrak{G}$ denote the topological groupoid of germs of $\mathcal{G}$. Since the G-action on G/P is faithful and quasi-analytic, and G/P is connected, we get a bijection $\Gamma_{U,x} \to \mathfrak{G}_x$, $\gamma \mapsto \operatorname{germ}(h_\gamma, x)$. For $\gamma \in \Gamma_{U,x}$, let $|\gamma|_{S,U,x} := |h_\gamma|_{E,x}$. Thus $|1|_{S,U,x} = 0$, and, if $\gamma \neq 1$, then $|\gamma|_{S,U,x}$ equals the minimum $n \in \mathbb{N}$ such that there are $i_1, \ldots, i_n \in \{1, \ldots, k\}$ with $\gamma = s_{i_n} \cdots s_{i_1}$ and $s_{i_m} \cdots s_{i_1} \cdot x \in U$ for all $m \in \{1, \ldots, n\}$. Moreover d_E on $\mathfrak{G}_x$ corresponds to the metric $d_{S,U,x}$ on $\Gamma_{U,x}$ given by
$$d_{S,U,x}(\gamma, \delta) = |\delta \gamma^{-1}|_{S,U,\gamma(x)} \,.$$
Observe that, for all $\gamma \in \Gamma_{U,x}$ and $\delta \in \Gamma_{U,\gamma \cdot x}$,

$$\delta\gamma \in \Gamma_{U,x} \,, \qquad |\delta\gamma|_{S,U,x} \leq |\gamma|_{S,U,x} + |\delta|_{S,U,\gamma \cdot x} \,, \tag{2}$$

$$\gamma^{-1} \in \Gamma_{U,\gamma \cdot x} \,, \qquad |\gamma|_{S,U,x} = |\gamma^{-1}|_{S,U,\gamma \cdot x} \,. \tag{3}$$

Assume that the $\mathcal{G}$-invariant set
$$A = \{\, x \in U \mid \operatorname{card} \mathfrak{G}_x < \infty \,\}$$
is open in T, and let $B = U \setminus A$.

Theorem 3.1. *With the above notation and conditions,*

- *either all infinite germ covers of the $\mathcal{G}$-orbits have polynomial growth with degree bounded by some common constant;*
- *or all infinite germ covers of the $\mathcal{G}$-orbits have exponential growth.*

Proof. Fix any open set V that meets all orbits such that $\overline{V} \subset U$.

Claim 1. For each finite subset $F \subset \Gamma$, we have

$$B \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma \cdot V \ .$$

Suppose that Claim 1 is false. Then there is some finite symmetric subset $F \subset \Gamma$ and some $x \in B$ such that $((\Gamma \setminus F) \cdot x) \cap V = \emptyset$. By the recurrence of E, there is some $N \in \mathbb{N}$ satisfying (1). Since $\Gamma_{U,x}$ is infinite because $x \in B$, it follows that there is some $\gamma \in \Gamma_{U,x} \setminus F$ such that

$$|\gamma|_{S,U,x} > N + \max\{\, |\epsilon|_{S,U,x} \mid \epsilon \in F \cap \Gamma_{U,x} \,\} \ . \tag{4}$$

By (1), there is some $h \in E^N$ such that

$$\gamma \cdot x \in h^{-1}(V \cap \operatorname{im} h) \ .$$

We have $h = h_\delta$ for some $\delta \in \Gamma$. Note that $\delta \in \Gamma_{U,\gamma \cdot x}$ and $|\delta|_{S,U,\gamma \cdot x} \leq N$. Hence

$$|\gamma|_{S,U,x} \leq |\delta\gamma|_{S,U,x} + |\delta^{-1}|_{S,U,\delta\gamma \cdot x} = |\delta\gamma|_{S,U,x} + |\delta|_{S,U,\gamma \cdot x} \leq |\delta\gamma|_{S,U,x} + N$$

by (2) and (3), obtaining that $\delta\gamma \notin F$ by (4). However $\delta\gamma \cdot x \in V$, obtaining a contradiction which completes the proof of Claim 1.

Claim 2. For each finite subset $F \subset \Gamma$, we have

$$\overline{B} \subset \bigcup_{\gamma \in \Gamma \setminus F} \gamma \cdot V \ .$$

Take a relatively compact open subset $U' \subset T$ such that $\overline{U} \subset U'$, and let $\mathcal{G}' = \mathcal{H}|_{U'}$. Define B' for $\mathcal{G}'$ like B for $\mathcal{G}$. Observe that $B \subset B'$. By using a recurrent system of compact generation of $\mathcal{H}$ on U', and applying Theorem 2.6 to the equivalence $\mathcal{G} \to \mathcal{G}'$ generated by $U \hookrightarrow U'$, we get that B' is the $\mathcal{G}'$-saturation of B. Since B is closed in U, it follows that B' is also closed on U', and therefore $\overline{B} \subset B'$. Then Claim 2 follows by applying Claim 1 to B'.

According to Claim 2, by increasing S if necessary, we can suppose that

$$\overline{B} \subset \bigcup_{i<j}(s_i \cdot V \cap s_j \cdot V) = \bigcup_{i<j}(s_i^{-1} \cdot V \cap s_j^{-1} \cdot V) \ . \tag{5}$$

If Γ is not nilpotent, then, by [5, Proposition 10.5], there are elements $t_1, \ldots, t_k$ in Γ, as close as desired to $s_1, \ldots, s_k$, respectively, which are free

generators of a free semi-group. By the compactness of $\overline{B}$, if $t_1, \ldots, t_k$ are close enough to $s_1, \ldots, s_k$, then (5) gives

$$\overline{B} \subset \bigcup_{i<j} (t_i^{-1} \cdot V \cap t_j^{-1} \cdot V) . \tag{6}$$

Now, we adapt the argument of the proof of [5, Lemma 10.6]. Let $\Gamma' \subset \Gamma$ be the subgroup generated by $t_1, \ldots, t_k$; thus $S' = \{t_1^{\pm 1}, \ldots, t_k^{\pm 1}\}$ is a symmetric set of generators of Γ', and $S \cup S'$ is a symmetric set of generators of Γ. With $E' = \{h_{t_1}^{\pm 1}, \ldots, h_{t_k}^{\pm 1}\}$, observe that $E \cup E'$ is a recurrent system of compact generation of $\mathcal{H}$ on U. Given $x \in B$, let $\mathsf{S}(n)$ be the sphere with center the identity element and radius $n \in \mathbb{N}$ in $\Gamma'_{U,x}$ with $d_{S',U,x}$. By (6) and since B is $\mathcal{G}$-invariant, for each $\gamma \in \mathsf{S}(n)$, there are indices $i < j$ such that $\gamma \cdot x \in t_i^{-1} \cdot V \cap t_j^{-1} \cdot V$. So the points $t_i \gamma \cdot x$ and $t_j \gamma \cdot x$ are in V, obtaining that $t_i \gamma, t_j \gamma \in \mathsf{S}(n+1)$. Moreover all elements obtained in this way from elements of $\mathsf{S}(n)$ are pairwise distinct because S' freely generate a free semigroup. Hence $\mathrm{card}(\mathsf{S}(n+1)) \geq 2\,\mathrm{card}(\mathsf{S}(n))$, giving $\mathrm{card}(\mathsf{S}(n)) \geq 2^n$. So $\Gamma'_{U,x}$ has exponential growth with $d_{S',U,x}$. Since $\Gamma'_{U,x} \subset \Gamma_{U,x}$ and $d_{S \cup S',U,x} \leq d_{S',U,x}$ on $\Gamma'_{U,x}$, it follows that $\Gamma_{U,x}$ also has exponential growth with $d_{S \cup S',U,x}$. So $\mathfrak{G}_x$ has exponential growth with $d_{E \cup E'}$, obtaining that $\mathfrak{G}_x$ has exponential growth with d_E by Corollary 2.7.

If Γ is nilpotent, then it has polynomial growth with respect to d_S, and this growth type dominates the growth type of $\Gamma_{U,x}$ with the metric $d_{S,U,x}$ because $d_S \leq d_{S,U,x}$ on $\Gamma_{U,x}$. $\qquad\square$

4. Growth of transversely homogeneous foliations

Suppose that G and P satisfy the conditions of Section 3, recall that a (*transversely homogeneous*) G/P-*foliation* is a smooth foliation $\mathcal{F}$ on a manifold M with the structure given by a defining cocycle (U_i, p_i, h_{ij}) *with values in* the G-manifold G/P in the sense that each $p_i(U_i)$ is an open subset of G/P, and the maps h_{ij} are restrictions of the canonical action of elements of G on G/P. Here, two defining cocycles with values in G/P define the same structure when both of them are included in some common defining cocycle with values in G/P.

Assume also that G/P is connected, and M is connected and compact. Then the holonomy pseudogroup of $\mathcal{F}$ is represented by the pseudogroup $\mathcal{H}$ generated by the action of some finitely generated subgroup $\Gamma \subset G$ on some Γ-invariant open subset $T \subset G/P$ [4, Theorem 1]; in particular, $\mathcal{H}$ is compactly generated. Let $U \subset T$ be a relatively compact open subset that meets all $\mathcal{H}$-orbits, and let $\mathcal{G} = \mathcal{H}|_U$. By the generalized Reeb stability theorem [6, Theorem 2.4.3], the union of leaves with compact holonomy covers is an open subset of M, and therefore the union of $\mathcal{G}$-orbits with finite germ covers is open in U by the remarks of Section 2.4. Then Theorem 1.1 is a direct consequence of Theorem 3.1 and the observations of Section 2.4.

Now suppose that $\mathcal{F}$ is taut. Let $m = \dim \mathcal{F}$ and let $\mathfrak{X}(\mathcal{F})$ be the family of smooth vector fields on M tangent to the leaves. To prove Corollary 1.2, we can assume the $\mathcal{F}$ is oriented by lifting $\mathcal{F}$ to its double cover of its orientations if necessary. Then the tautness condition means that there is an m-form χ on M whose restriction to the leaves is positive and such that $i_{X_1} \ldots i_{X_m} d\chi = 0$ if $X_i \in \mathfrak{X}(\mathcal{F})$ for all $i \in \{1, \ldots, m\}$ [17, 18, 10]. Besides the quasi-analyticity of the holonomy pseudogroup, Pereira's global stability theorem [16, Theorem 1 and Remark 6] requires the existence of a closed m-form on M whose restriction to the leaves is positive, but indeed its proof only needs the existence of a form satisfying the conditions of the above χ (see the proof of [16, Lemma 3]). So, according to that global stability theorem, all holonomy covers of the leaves are compact if one of them is compact, and therefore Corollary 1.2 follows from Theorem 1.1.

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Departamento de Xeometría e Topoloxía, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
   E-mail address: jesus.alvarez@usc.es

Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, ul. prof. Stanisława Łojasiewicza 6, 30-348 Kraków, Poland
   E-mail address: robert.wolak@im.uj.edu.pl